Proof of some conjectured formulas for $\frac{1}{\pi}$ by Z.W.Sun.

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Recently Z.W.Sun found over hundred conjectured formulas for $\frac{1}{\pi}$. Many of them were proved by H.H.Chan, J.Wan and W.Zudilin (see [3], [9]). Here we show that several other formulas in [6] are simple transformations of known formulas for $\frac{1}{\pi}$, most of them due to Ramanujan. E.g. the following monstrous formula (not in [6])

$$\sum_{n=0}^{\infty} A_n P(n) \frac{1}{262537412640769728^n} = \frac{1380398151109206244 0689}{\pi\sqrt{163}}$$

where

$$P(n) = 41299228622713244768 05 + 16564777691765267456000n$$

and

$$A_n = 1728^n \sum_{k=0}^{n} \binom{-1/12}{k}\binom{-7/12}{k}\binom{-5/12}{n-k}\binom{-11/12}{n-k}$$

is a transformation of Chudnovsky's formula

$$\sum_{n=0}^{\infty} (-1)^n a_n (13591409 + 545140134n) \frac{1}{640320^{3n}} = \frac{53360\sqrt{640320}}{12\pi}$$

where

$$a_n = \binom{2n}{n}\binom{3n}{n}\binom{6n}{3n}$$

The transformation is

$$\sum_{n=0}^{\infty} A_n x^n = \frac{1}{\sqrt{1-1728x}} \sum_{n=0}^{\infty} a_n \left(-\frac{x}{1-1728x}\right)^n$$

General Transformation.
Assume that we have a Ramanujan-like formula

$$\sum_{n=0}^{\infty} a_n (a+bn) x_0^n = \frac{1}{\pi}$$

We make the substitution

$$\sum_{n=0}^{\infty} A_n x^n = \frac{1}{\sqrt{1-Mx}} \sum_{n=0}^{\infty} a_n \left(-\frac{x}{1-Mx}\right)^n$$

Proposition 1. Let

$$w_0 = -\frac{x_0}{1-Mx_0}$$

Then we have the formula

$$\sum_{n=0}^{\infty} A_n(A+Bn)w_0^n = \frac{1}{\pi}$$

where

$$A = \left\{\frac{1}{2}bMx_0 + a(1-Mx_0)\right\}(1-Mx_0)^{-3/2}$$

and

$$B = b(1-Mx_0)^{-3/2}$$

Proof: The transformation above is an involution, e.g. we also have

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{\sqrt{1-Mx}} \sum_{n=0}^{\infty} A_n \left(-\frac{x}{1-Mx}\right)^n$$

Let $\theta = x\dfrac{d}{dx}$. Then

$$\sum_{n=0}^{\infty} a_n(a+bn)x^n = (a+b\theta)\sum_{n=0}^{\infty} a_n x^n$$

$$= (a+b\theta)\left\{\frac{1}{\sqrt{1-Mx}} \sum_{n=0}^{\infty} A_n \left(-\frac{x}{1-Mx}\right)^n\right\}$$

$$= \sum_{n=0}^{\infty}(-1)^n A_n (a+b\theta)\left\{\frac{x^n}{(1-Mx)^{n+1/2}}\right\}$$

$$= \sum_{n=0}^{\infty} A_n \left\{\frac{a+bn}{(1-Mx)^{1/2}} + \frac{(n+\frac{1}{2})bx}{(1-Mx)^{3/2}}\right\}\left(-\frac{x}{1-Mx}\right)^n$$

Substituting $x = x_0$ we are done.

The example in the introduction is the case $s = \dfrac{1}{6}$ and $M = 1728$ of the hypergeometric case

$$a_n = M^n \frac{(1/2)_n (s)_n (1-s)_n}{n!^3}$$

Proving the transformation is a Maple exercise in each of the cases $s = \dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{6}$. E.g. in the case $s = \dfrac{1}{6}$ one shows that both sides satisfy the differential equation

$$y''' + \frac{3(1-3456x)}{x(1-1728x)}y'' + \frac{1-11856x+20155392x^2}{x^2(1-1728x)^2}y' - \frac{24(31-93312x)}{x^2(1-1728x)^2} = 0$$

and checks that the first four coefficients agree.

The case s =$\frac{1}{3}$.

Here we have $M = 108$ and

$$a_n = 108^n \frac{(1/2)_n(1/3)_n(2/3)_n}{n!^3} = \binom{2n}{n}^2 \binom{3n}{n}$$

with

$$A_n = 108^n \sum_{k=0}^{n} \binom{-2/3}{k}\binom{-1/6}{k}\binom{-1/3}{n-k}\binom{-5/6}{n-k}$$

In the table below the fomula

$$\sum_{n=0}^{\infty} a_n(a+bn)x_0^n = \frac{1}{\pi}$$

is transformed to

$$\sum_{n=0}^{\infty} A_n(A+Bn)w_0^n = \frac{1}{\pi}$$

# in [6]	x_0	a	b	w_0	A	B
4.16	$-\frac{1}{192}$	$\frac{\sqrt{3}}{4}$	$\frac{5\sqrt{3}}{4}$	$\frac{1}{300}$	$\frac{\sqrt{3}}{50}$	$\frac{16\sqrt{3}}{25}$
4.18	$-\frac{1}{1728}$	$\frac{7\sqrt{3}}{36}$	$\frac{17\sqrt{3}}{12}$	$\frac{1}{1836}$	$\frac{11\sqrt{51}}{306}$	$\frac{48\sqrt{51}}{153}$
4.20	$-\frac{1}{8640}$	$\frac{\sqrt{15}}{12}$	$\frac{3\sqrt{15}}{4}$	$\frac{1}{8748}$	$\frac{85\sqrt{3}}{486}$	$\frac{400\sqrt{3}}{243}$
4.21	$-\frac{1}{108\cdot 2^{10}}$	$\frac{53\sqrt{3}}{288}$	$\frac{205\sqrt{3}}{96}$	$\frac{1}{110700}$	$\frac{527\sqrt{123}}{18450}$	$\frac{3072\sqrt{123}}{9225}$
-	$-\frac{1}{108\cdot 3024}$	$\frac{13\sqrt{7}}{108}$	$\frac{55\sqrt{7}}{36}$	$\frac{1}{326700}$	$\frac{9989\sqrt{3}}{54450}$	$\frac{127008\sqrt{3}}{54450}$
4.22	$-\frac{1}{108\cdot 500^2}$	$\frac{827\sqrt{3}}{4500}$	$\frac{4717\sqrt{3}}{1500}$	$\frac{1}{27000108}$	$\frac{97659\sqrt{267}}{4500018}$	$\frac{1500000\sqrt{267}}{4500018}$
4.17	$\frac{1}{1458}$	$\frac{8}{27}$	$\frac{20}{9}$	$-\frac{1}{1350}$	$\frac{52\sqrt{3}}{225}$	$\frac{36\sqrt{3}}{25}$
4.19	$\frac{1}{27\cdot 125}$	$\frac{8\sqrt{3}}{45}$	$\frac{22\sqrt{3}}{45}$	$-\frac{1}{3267}$	$\frac{100\sqrt{15}}{1089}$	$\frac{250\sqrt{15}}{363}$
4.14	$-\frac{1}{27}$	$\frac{4\sqrt{3}}{9}$	$\frac{5\sqrt{3}}{3}$	$\frac{1}{135}$	$-\frac{2\sqrt{15}}{45}$	$\frac{\sqrt{15}}{15}$

The last formula from columns 2-4 is divergent but results in the following supercongruence

$$\sum_{n=0}^{p-1} a_n(4+15n)\frac{1}{(-27)^n} \equiv 4p\left(\frac{-3}{p}\right) \mod p^3$$

conjectured by Z.W.Sun in [8].

The case $\mathbf{s = \frac{1}{4}}$.

Here we have $M = 256$ and

$$a_n = 256^n \frac{(1/2)_n (1/4)_n (3/4)_n}{n!^3} = \binom{2n}{n}^2 \binom{4n}{2n}$$

with

$$A_n = 256^n \sum_{k=0}^{n} \binom{-1/8}{k} \binom{-5/8}{k} \binom{-3/8}{n-k} \binom{-7/8}{n-k}$$

# in [6]	x_0	a	b	w_0	A	B
4.23	$-\frac{1}{1024}$	$\frac{3}{8}$	$\frac{5}{2}$	$\frac{1}{1280}$	$\frac{\sqrt{5}}{20}$	$\frac{4\sqrt{5}}{5}$
-	$-\frac{1}{63^2}$	$\frac{8\sqrt{7}}{63}$	$\frac{65\sqrt{7}}{63}$	$\frac{1}{4225}$	$\frac{392\sqrt{7}}{4225}$	$\frac{3969\sqrt{7}}{4225}$
4.27	$-\frac{1}{3\cdot 2^{12}}$	$\frac{3\sqrt{3}}{16}$	$\frac{7\sqrt{3}}{4}$	$\frac{1}{12544}$	$\frac{57}{196}$	$\frac{144}{49}$
4.29	$-\frac{1}{288^2}$	$\frac{23}{72}$	$260\frac{65}{18}$	$\frac{1}{83200}$	$\frac{113\sqrt{13}}{1300}$	$\frac{324\sqrt{13}}{325}$
-	$-\frac{1}{1280\cdot 72^2}$	$\frac{41\sqrt{5}}{288}$	$644\frac{161\sqrt{5}}{72}$	$\frac{1}{6635776}$	$\frac{32995}{103684}$	$\frac{518400}{103684}$
-	$-\frac{1}{14112^2}$	$\frac{1123}{3528}$	$\frac{5365}{882}$	$\frac{1}{199148800}$	$\frac{162833\sqrt{37}}{3111700}$	$\frac{3111696\sqrt{37}}{3111700}$
4.26	$\frac{1}{648}$	$\frac{2}{9}$	$\frac{14}{9}$	$-\frac{1}{392}$	$\frac{46}{49}$	$\frac{162}{49}$
4.25	$\frac{1}{48^2}$	$\frac{\sqrt{3}}{6}$	$\frac{4\sqrt{3}}{3}$	$-\frac{1}{2048}$	$\frac{3\sqrt{3}}{16}$	$\frac{9\sqrt{3}}{8}$
4.28	$\frac{1}{144^2}$	$\frac{2\sqrt{2}}{9}$	$\frac{20\sqrt{2}}{9}$	$-\frac{1}{20480}$	$\frac{17\sqrt{10}}{80}$	$\frac{81\sqrt{10}}{40}$
4.30	$\frac{1}{784^2}$	$\frac{9\sqrt{3}}{49}$	$\frac{120\sqrt{3}}{49}$	$-\frac{1}{614400}$	$\frac{361\sqrt{2}}{1600}$	$\frac{2401\sqrt{2}}{800}$
4.31	$\frac{1}{16^2\cdot 99^2}$	$\frac{19\sqrt{11}}{198}$	$\frac{140\sqrt{11}}{99}$	$-\frac{1}{2508800}$	$\frac{1331\sqrt{22}}{19600}$	$\frac{9801\sqrt{22}}{9800}$
-	$\frac{1}{16^2\cdot 99^4}$	$\frac{2206\sqrt{2}}{9801}$	$\frac{52780\sqrt{2}}{9801}$	$-\frac{1}{24591257600}$	$\frac{8029841\sqrt{58}}{192119200}$	$\frac{192119202\sqrt{58}}{192119200}$
4.24	$-\frac{1}{144}$	$\frac{\sqrt{3}}{3}$	$\frac{5\sqrt{3}}{3}$	$\frac{1}{400}$	$-\frac{3\sqrt{3}}{25}$	$\frac{9\sqrt{3}}{25}$

The last Ramanujan-like formula is divergent (for a proof see Guillera [2]) but leads

to the conjectured supercongruence (already in [8])

$$\sum_{n=0}^{p-1} a_n(1+5n)\frac{1}{(-144)^n} \equiv p\left(\frac{-3}{p}\right) \mod p^3$$

The case $\mathbf{s = \frac{1}{6}}$.
Here $M = 1728$ and

$$a_n = 1728^n \frac{(1/2)_n(1/6)_n(5/6)_n}{n!^3} = \binom{2n}{n}\binom{3n}{n}\binom{6n}{3n}$$

with

$$A_n = 1728^n \sum_{k=0}^{n} \binom{-1/12}{k}\binom{-7/12}{k}\binom{-5/12}{n-k}\binom{-11/12}{n-k}$$

# in [6]	x_0	a	b	w_0	A	B
4.33	$-\frac{1}{15^3}$	$\frac{8\sqrt{15}}{75}$	$\frac{21\sqrt{15}}{25}$	$\frac{1}{5103}$	$-\frac{8\sqrt{7}}{189}$	$\frac{125\sqrt{7}}{189}$
4.36	$-\frac{1}{32^3}$	$\frac{15\sqrt{2}}{64}$	$\frac{77\sqrt{2}}{32}$	$\frac{1}{34496}$	$\frac{39\sqrt{11}}{539}$	$\frac{512\sqrt{11}}{539}$
-	$-\frac{1}{3\cdot 160^3}$	$\frac{93\sqrt{30}}{1600}$	$\frac{759\sqrt{30}}{800}$	$\frac{1}{12289728}$	$\frac{11751\sqrt{3}}{64009}$	$\frac{192000\sqrt{3}}{64009}$
4.35	$-\frac{1}{96^3}$	$\frac{25\sqrt{6}}{192}$	$\frac{57\sqrt{6}}{32}$	$\frac{1}{886464}$	$\frac{37\sqrt{19}}{513}$	$\frac{512\sqrt{19}}{513}$
-	$-\frac{1}{960^3}$	$\frac{263\sqrt{15}}{3200}$	$\frac{2709\sqrt{15}}{1600}$	$\frac{1}{884737728}$	$\frac{248853\sqrt{43}}{512001}$	$\frac{512000\sqrt{43}}{512001}$
-	$-\frac{1}{5280^3}$	$\frac{10177\sqrt{330}}{580800}$	$\frac{43617\sqrt{330}}{96800}$	$\frac{1}{147197953728}$	$\frac{3312613\sqrt{67}}{85184001}$	$\frac{85184000\sqrt{67}}{85184001}$
4.34	$\frac{1}{20^3}$	$\frac{3\sqrt{5}}{25}$	$\frac{28\sqrt{5}}{25}$	$-\frac{1}{6272}$	$\frac{24\sqrt{2}}{49}$	$\frac{125\sqrt{2}}{49}$
4.32	$\frac{4}{60^3}$	$\frac{2\sqrt{15}}{25}$	$\frac{22\sqrt{15}}{25}$	$-\frac{1}{52272}$	$\frac{26}{121}$	$\frac{250}{121}$
4.37	$\frac{1}{66^3}$	$\frac{20\sqrt{33}}{363}$	$252\frac{84\sqrt{33}}{121}$	$-\frac{1}{285768}$	$\frac{436}{1323}$	$\frac{5324}{1323}$
-	$\frac{1}{255^3}$	$\frac{144\sqrt{255}}{7225}$	$\frac{2394\sqrt{255}}{7225}$	$-\frac{1}{16579647}$	$\frac{73936\sqrt{7}}{614061}$	$\frac{1228250\sqrt{7}}{614061}$
-	$-\frac{1}{640320^3}$	$\frac{13591409\sqrt{10005}}{227897059584000}$	$\frac{90856689\sqrt{10005}}{37982843264000}$			

We have deleted the formula obtained from Chudnovsky's formula since it is in the Introduction.

So far we have only considered Ramanujan series with rational x_0, found in [1]. We give one example in case $s = \frac{1}{3}$ with $x_0 = \frac{1}{72}(7\sqrt{3} - 12)$

$$\sum_{n=0}^{\infty} a_n(1+(5+\sqrt{3})n)\left(\frac{7\sqrt{3}-12}{72}\right)^n = \frac{2+\sqrt{3}}{\pi}$$

giving

$$\sum_{n=0}^{\infty} A_n\left\{2(27\sqrt{3}-41)+8(5+\sqrt{3})n\right\}\left(\frac{15-14\sqrt{3}}{66^2}\right)^n = \frac{254-134\sqrt{3}}{\pi}$$

In the paper [7] by Z.W.Sun there are some formulas for $\frac{1}{\pi}$ which are special cases of identities for the hypergeometric function

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n$$

Thus Theorem 1.1 (i) in [7] is the special cases $s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ of

Proposition 2:

We have

$$\sum_{n=0}^{\infty} n(\frac{1}{2})^n \sum_{k=0}^{n} \frac{(s)_k(1-s)_k}{k!^2}\frac{(s)_{n-k}(1-s)_{n-k}}{(n-k)!^2} = \frac{2}{\pi}\sin(\pi s)$$

Proof: Let

$$f(x) = F(s,1-s;1;x)^2$$
$$= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{(s)_k(1-s)_k}{k!^2}\frac{(s)_{n-k}(1-s)_{n-k}}{(n-k)!^2}$$

Then

$$\theta f(x) = \sum_{n=0}^{\infty} n x^n \sum_{k=0}^{n} \frac{(s)_k(1-s)_k}{k!^2}\frac{(s)_{n-k}(1-s)_{n-k}}{(n-k)!^2}$$
$$= 2s(1-s)x\cdot F(s,1-s;1;x)\cdot F(1+s,2-s;2;x)$$

Put $x = \frac{1}{2}$ and use the evaluation

$$F(a,b;\frac{a+b+1}{2};\frac{1}{2}) = \sqrt{\pi}\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}$$

We obtain

$$2\frac{1}{2}s(1-s)\pi\frac{\Gamma(1)}{\Gamma(\frac{1}{2}+\frac{s}{2})\Gamma(1-\frac{s}{2})}\frac{\Gamma(2)}{\Gamma(1+\frac{s}{2})\Gamma(\frac{3}{2}-\frac{s}{2})}$$
$$= s(1-s)\pi\frac{1}{\Gamma(\frac{1}{2}-\frac{s}{2})\frac{s}{2}\Gamma(\frac{s}{2})}\frac{1}{\Gamma(\frac{1}{2}+\frac{s}{2})(\frac{1}{2}-\frac{s}{2})\Gamma(\frac{1}{2}-\frac{s}{2})}$$

$$= 4\pi \frac{\sin(\frac{\pi s}{2})}{\pi} \frac{\sin(\pi(\frac{1}{2} - \frac{s}{2}))}{\pi} = \frac{4}{\pi} \sin(\frac{\pi s}{2}) \cos(\frac{\pi s}{2}) = \frac{2}{\pi} \sin(\pi s)$$

Some other transformations.

We start with proving Conjecture 4 in [6]. We have

Proposition 3. Let

$$A_n = \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \binom{-s}{k} \binom{-(1-s)}{n-k}$$

Then the following formula is valid

$$\sum_{n=0}^{\infty} A_n x^n = \frac{1}{\sqrt{1+4x}} \ {}_3F\ {}_2(1/2, s, 1-s; 1, 1; -\frac{4x^2}{1+4x})$$

Classical Proof:

We first note the identities

$$\binom{2k}{k} = 4^k \frac{(1/2)_k}{k!}$$

$$\binom{-s}{k} = (-1)^k \frac{(s)_k}{k!}$$

where $(a)_0 = 1$ and

$$(a)_k = a(a+1)...(a+k-1) \ \text{ for } \ k > 0$$

We get

$$L = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \binom{-s}{k} \binom{-(1-s)}{n-k} x^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(1/2)_k (s)_k}{k!^2} \frac{(1/2)_{n-k} (1-s)_{n-k}}{(n-k)!^2} (-4x)^n$$

$$= F(1/2, s; 1; -4x) F(1/2, 1-s; 1, -4x)$$

Now Euler's identity

$$F(a, b; , c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x)$$

leads to

$$F(1/2, 1-s; 1, -4x) = (1+4x)^{-1/2+s} F(1/2, s; 1; -4x)$$

Hence

$$L = \frac{1}{\sqrt{1+4x}} \left\{ (1+4x)^{s/2} F(1/2, s; 1; -4x) \right\}^2$$

Now we use the following identity (see [2], p.176, Exercise 1b)

$$F(2a,b;2b;x) = (1-x)^{-a} F(a, b-a; b+1/2; \frac{x^2}{4(x-1)}$$

with $a = s/2, b = 1/2$ to get

$$F(s,1/2;1;-4x) = (1+4x)^{-s/2} F(s/2,(1-s)/2;1;-\frac{4x^2}{1+4x})$$

Finally Clausen's identity

$$F(a,b;a+b+1/2;x)^2 = F(2a,2b,a+b;a+b+1/2,2a+2b;x)$$

gives

$$L = \frac{1}{\sqrt{1+4x}} F(1/2,s,1-s;1,1;-\frac{4x^2}{1+4x})$$

and the proof is finished.

Maple Proof:

Using Maple one verifies that both sides satisfy the differential equation

$$y''' + \frac{3(1+8x)}{x(1+4x)} y'' + \frac{1+28x+(108+16s-16s^2)x^2}{x^2(1+4x)^2} y' + \frac{2(1+(6+8s-8s^2)x)}{x^2(1+4x)^2} y = 0$$

Then we check that the first terms in the power series solutions agree.

Proposition 4. Let

$$a_n = \frac{(1/2)_n (s)_n (1-s)_n}{n!^3}$$

Given a formula for $\frac{1}{\pi}$ of Ramanujan type

$$\sum_{n=0}^{\infty} a_n (a+bn) x_0^n = \frac{1}{\pi}$$

Let

$$w_0 = \frac{1}{2}(-x_0 \pm \sqrt{x_0^2 - x_0})$$

Then the transformation above gives the formulas

$$\sum_{n=0}^{\infty} A_n (A+Bn) w_0^n = \frac{1}{\pi}$$

where

$$A = \sqrt{1+4w_0} \left\{ a + \frac{bw_0}{1+2w_0} \right\}$$

$$B = \frac{b(1+4w_0)^{3/2}}{2(1+2w_0)}$$

Proof: We have

$$\sum_{n=0}^{\infty} A_n w^n = \frac{1}{\sqrt{1+4w}} \sum_{n=0}^{\infty} a_n \left(-\frac{w^2}{1+4w}\right)^n$$

Take $A + B\theta$ on both sides ($\theta = w\frac{d}{dw}$)

$$\sum_{n=0}^{\infty} A_n (A+Bn) w^n = \sum_{n=0}^{\infty} a_n \left\{ \frac{A}{\sqrt{1+4w}} + \frac{2B}{(1+4w)^{3/2}}(-w+(2w+1)n) \right\} \left(-\frac{w^2}{1+4w}\right)^n$$

Now put $w = w_0$ so $-\frac{w_0^2}{1+4w_0} = x_0$ and the right hand is $\sum_{n=0}^{\infty} a_n(a+bn)x_0^n$. We get

$$a = \frac{A}{\sqrt{1+4w_0}} - \frac{2Bw_0}{(1+4w_0)^{3/2}}$$

$$b = \frac{2B(2w_0+1)}{(1+4w_0)^{3/2}}$$

and solving for A and B we are done.

$s = \frac{1}{2}$

x_0	a	b	w_0	A	B
-1	$\frac{1}{2}$	2	$\frac{1}{2}(1-\sqrt{2})$	$\frac{-3+2\sqrt{2}}{2}$	$\frac{-4+3\sqrt{2}}{2}$
$-\frac{1}{8}$	$\frac{\sqrt{2}}{4}$	$\frac{3\sqrt{2}}{2}$	$\frac{1}{16}(1\pm 3)$	$\frac{1}{2}(1\pm 1)$	$\frac{1}{4}(5\pm 3)$

$s = \frac{1}{3}$

x_0	a	b	w_0	A	B
$-\frac{9}{16}$	$\frac{\sqrt{3}}{4}$	$\frac{5\sqrt{3}}{4}$	$\frac{3}{32}(3\pm 5)$	$\frac{\sqrt{3}}{32}(19\pm 21)$	$\frac{\sqrt{3}}{16}(17\pm 15)$
$-\frac{1}{16}$	$\frac{7\sqrt{3}}{36}$	$\frac{17\sqrt{3}}{12}$	$\frac{1}{32}(1\pm\sqrt{17})$	$\frac{17\sqrt{51}\pm 65\sqrt{3}}{288}$	$\frac{9\sqrt{51}\pm 17\sqrt{3}}{48}$
$-\frac{1}{80}$	$\frac{\sqrt{15}}{12}$	$\frac{3\sqrt{15}}{4}$	$\frac{1}{160}(1\pm 9)$	$\frac{\sqrt{3}}{96}(19\pm 11)$	$\frac{\sqrt{3}}{48}(41\pm 9)$
$-\frac{1}{1024}$	$\frac{53\sqrt{3}}{288}$	$\frac{205\sqrt{3}}{96}$	$\frac{1}{2048}(1\pm 5\sqrt{41})$	$\frac{533\sqrt{123}\pm 721\sqrt{3}}{18432}$	$\frac{513\sqrt{123}\pm 205\sqrt{3}}{3072}$
$-\frac{1}{3024}$	$\frac{13\sqrt{7}}{108}$	$\frac{55\sqrt{7}}{36}$	$\frac{1}{6048}(1\pm 55)$	$\frac{\sqrt{3}}{7776}(1433\pm 191)$	$\frac{\sqrt{3}}{1296}(1513\pm 55)$
$-\frac{1}{500^2}$	$\frac{827\sqrt{3}}{4500}$	$\frac{4717\sqrt{3}}{1500}$	$\frac{1}{500000}(1\pm 53\sqrt{89})$	$\frac{17533\sqrt{267}\pm 3161\sqrt{3}}{900000}$	$\frac{125001\sqrt{267}\pm 4717\sqrt{3}}{750000}$

$s = \frac{1}{4}$

x_0	a	b	w_0	A	B
$-\frac{1}{4}$	$\frac{3}{8}$	$\frac{5}{2}$	$\frac{1}{8}(1 \pm \sqrt{5})$	$\frac{\pm 13 + 5\sqrt{5}}{16}$	$\frac{\pm 5 + 3\sqrt{5}}{4}$
$-(\frac{16}{63})^2$	$\frac{8\sqrt{7}}{63}$	$\frac{65\sqrt{7}}{63}$	$\frac{8}{63^2}(16 \pm 65)$	$\frac{8\sqrt{7}}{49}(1 \pm 1)$	$\frac{\sqrt{7}}{7038}(4481 \pm 2080)$
$-\frac{1}{48}$	$\frac{\sqrt{3}}{16}$	$\frac{7\sqrt{3}}{12}$	$\frac{1}{96}(1 \pm 7)$	$\frac{1}{192}(23 \pm 17)$	$\frac{1}{48}(25 \pm 7)$
$-\frac{1}{324}$	$\frac{23}{72}$	$\frac{65}{18}$	$\frac{1}{648}(1 \pm 5\sqrt{13})$	$\frac{\pm 17 + 13\sqrt{13}}{144}$	$\frac{\pm 65 + 163\sqrt{13}}{324}$
$-\frac{1}{5 \cdot 72^2}$	$\frac{41\sqrt{5}}{288}$	$\frac{161\sqrt{5}}{72}$	$\frac{1}{51840}(1 \pm 161)$	$\frac{1}{6912}(2201 \pm 121)$	$\frac{1}{5184}(12961 \pm 161)$
$-\frac{1}{882^2}$	$\frac{1123}{3528}$	$\frac{5365}{882}$	$\frac{1}{1555848}(1 \pm 145\sqrt{37})$	$\frac{\pm 439 + 6031\sqrt{37}}{115248}$	$\frac{\pm 5365 + 388963\sqrt{37}}{777924}$

$s = \frac{1}{6}$

x_0	a	b	w_0	A	B
$-(\frac{4}{5})^3$	$\frac{8\sqrt{15}}{75}$	$\frac{21\sqrt{15}}{25}$	$\frac{32 \pm 12\sqrt{21}}{125}$	$\frac{168\sqrt{7} \pm 316\sqrt{3}}{375}$	$\frac{253\sqrt{7} \pm 336\sqrt{3}}{250}$
$-(\frac{3}{8})^3$	$\frac{15\sqrt{2}}{64}$	$\frac{77\sqrt{2}}{32}$	$\frac{27 \pm 21\sqrt{33}}{1024}$	$\frac{33\sqrt{11} \pm 69\sqrt{3}}{256}$	$\frac{283\sqrt{11} \pm 231\sqrt{3}}{512}$
$-\frac{1}{512}$	$\frac{25\sqrt{6}}{192}$	$\frac{57\sqrt{6}}{32}$	$\frac{1 \pm 3\sqrt{57}}{1024}$	$\frac{57\sqrt{19} \pm 49\sqrt{3}}{768}$	$\frac{257\sqrt{19} \pm 57\sqrt{3}}{512}$
$-\frac{9}{40^3}$	$\frac{93\sqrt{30}}{1600}$	$\frac{759\sqrt{30}}{800}$	$\frac{9 \pm 759}{128000}$	$\frac{\sqrt{3}(5889 \pm 639)}{32000}$	$\frac{\sqrt{3}(96027 \pm 2277)}{64000}$
$-\frac{1}{80^3}$	$\frac{263\sqrt{15}}{3200}$	$\frac{2709\sqrt{15}}{1600}$	$\frac{1 \pm 63\sqrt{129}}{1024000}$	$\frac{12427\sqrt{43} \pm 743\sqrt{3}}{256000}$	$\frac{256001\sqrt{43} \pm 2709\sqrt{3}}{512000}$
$-\frac{1}{440^3}$	$\frac{10177\sqrt{330}}{580800}$	$\frac{43617\sqrt{330}}{96800}$	$\frac{1 \pm 651\sqrt{201}}{170368000}$	$\frac{4968921\sqrt{67} \pm 35257\sqrt{3}}{127776000}$	$\frac{42592001\sqrt{67} \pm 43617\sqrt{3}}{85184000}$
$-\frac{1}{53360^3}$	$\frac{13591409\sqrt{10005}}{227897059584000}$	$\frac{90856689\sqrt{10005}}{37982843264000}$	$\frac{1 \pm 557403\sqrt{489}}{303862746112000}$	$\frac{5681919113121\sqrt{163} \pm 71540369\sqrt{3}}{12160587099402240000}$	$\frac{75965686528001\sqrt{163} \pm 90856689\sqrt{3}}{8107058066268160000}$

This takes care of formulas 4.2-4.13 except 4.7 which comes from a divergent series with $x_0 = -\frac{16}{9}$. Note that we find a new formula with rational w_0 for $s = \frac{1}{6}$.

Remark: Formula (4.11) in [6] is false. The right hand side should be $\frac{162\sqrt{7}}{343\pi}$.

Formula 4.1 is of different kind. It is a special case of

Proposition 5.
We have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{-s}{k}^2\binom{-(1-s)}{n-k}^2 x^n = \frac{1}{1-x}F(1/2,s,1-s;1,1;-\frac{4x}{(1-x)^2})$$

Classical Proof:
The left hand side is

$$L=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(s)_k^2}{k!^2}\frac{(1-s)_{n-k}^2}{(n-k)!^2}x^n = F(s,s;,1;x)F(1-s,1-s;1;x)$$

$$= \frac{1}{1-x}F(s,1-s;1;\frac{x}{x-1})^2$$

after using Pfaff's identity twice

$$F(a,b;c;x)=(1-x)^{-a}F(a,c-b;c;\frac{x}{x-1})$$

Now we use

$$F(2a,2b;a+b+1/2;x)=F(a,b;a+b+1/2;4x(1-x))$$

again to get

$$L=\frac{1}{1-x}F(s/2,(1-s)/2;1;-\frac{4x}{(1-x)^2})^2$$
$$=\frac{1}{1-x}F(1/2,s,1-s;1,1;-\frac{4x}{(1-x)^2})$$

by Clausen's identity.

Maple Proof:
Both sides satisfy

$$y'''+\frac{3(2-5x)}{x(1-x)}y''+\frac{1-(10+s-s^2)x+(12+s-s^2)x^2}{x^2(1-x)^2}y'-\frac{1}{2}\frac{2+s-s^2-(6+3s-3s^2)x)}{x^2(1-x)^2}y=0$$

Then we check the first terms in the power series.

Let

$$a_n=\frac{(1/2)_n(s)_n(1-s)_n}{n!^3}$$

and

$$A_n=\sum_{k=0}^{n}\binom{-s}{k}^2\binom{-(1-s)}{n-k}^2$$

Then copying the proof of Proposition 4 we get for every formula

$$\sum_{n=0}^{\infty}a_n(a+bn)x_0^n=\frac{1}{\pi}$$

a new formula

$$\sum_{n=0}^{\infty}A_n(A+Bn)w_0^n=\frac{1}{\pi}$$

where

$$w_0=1-\frac{2}{x_0}(1-\sqrt{1-x_0})$$

$$A = (1-w_0)(a - \frac{bw_0}{1+w_0})$$
$$B = \frac{b(1-w_0)^2}{1+w_0}$$
We give only the rational w_0

s	x_0	a	b	w_0	A	B
$\frac{1}{3}$	$-\frac{9}{16}$	$\frac{\sqrt{3}}{4}$	$\frac{5\sqrt{3}}{4}$	$\frac{1}{9}$	$\frac{\sqrt{3}}{9}$	$\frac{8\sqrt{3}}{9}$
$\frac{1}{4}$	$-(\frac{16}{63})^2$	$\frac{8\sqrt{7}}{63}$	$\frac{65\sqrt{7}}{63}$	$\frac{1}{64}$	$\frac{7\sqrt{7}}{64}$	$\frac{63\sqrt{7}}{64}$
$\frac{1}{4}$	$\frac{32}{81}$	$\frac{2}{9}$	$\frac{14}{9}$	$-\frac{1}{8}$	$\frac{1}{2}$	$\frac{9}{4}$

The formulas (2.2)-(2.4) in [6] due to the twin brother Z.H.Sun are special cases of the following

Proposition 6: We have

$$F(s,1-s;1;\frac{1}{2}(1-\sqrt{1-x}))^2 = F(\frac{1}{2},s,1-s;1,1;x)$$

Classical Proof:

Solving for x we have the equivalent statement

$$F(s,1-s;1;w)^2 = F(\frac{1}{2},s,1-s;1,1;4w(1-w))$$

Using (formula 3.1.3, p.125 in [2])

$$F(2a,2b;,a+b+1/2;w) = F(a,b;a+b+1/2;4w(1-w))$$

we get

$$F(s,1-s;1;w)^2 = F(s/2,(1-s)/2;1;4w(1-w))^2$$

and finish by Clausen's identity.

Maple Proof:

One verifies that both sides satisfy the differential equation

$$y''' + \frac{3}{2}\frac{2-3x}{x(1-x)}y'' + \frac{1-(3+s-s^2)x}{x^2(1-x)}y' - \frac{1}{2}\frac{s(1-s)}{x^2(1-x)}y = 0$$

One expands both sides in power series and checks the first few coefficients.

Theorem 1.3 in [7] is a special case of the following transformation.

Let

$$A_n = \sum_{k=0}^{n} \frac{(s)_k(1-s)_k(s)_{n-k}(1-s)_{n-k}}{k!^2(n-k)!^2}$$

so

$$F(s,1-s;1;x)^2 = \sum_{n=0}^{\infty} A_n x^n$$

so we have the following result.

Proposition 7.
Assume we have a formula

$$\sum_{n=0}^{\infty} \frac{(1/2)_n(s)_n(1-s)_n}{n!^3}(a+bn)x_0^n = \frac{1}{\pi}$$

Then we have

$$\sum_{n=0}^{\infty} A_n(A+Bn)w_0^n = \frac{1}{\pi}$$

where

$$w_0 = \frac{1}{2}(1-\sqrt{1-x_0})$$

and

$$A = a$$
$$B = b\frac{1-w_0}{1-2w_0}$$

Proof: Let $\theta = x\dfrac{d}{dx}$ and $a_n = \dfrac{(1/2)_n(s)_n(1-s)_n}{n!^3}$. Then

$$\sum_{n=0}^{\infty} a_n(a+bn)x^n = (a+b\theta)\sum_{n=0}^{\infty} a_n x^n$$

$$= (a+b\theta)\sum_{n=0}^{\infty} A_n(\frac{1}{2}(1-\sqrt{1-x}))^n = \sum_{n=0}^{\infty} A_n(a+\frac{bn}{2\sqrt{1-x}(1-\sqrt{1-x})})(\frac{1}{2}(1-\sqrt{1-x}))^n$$

Putting $x = x_0$ we get

$$A = a$$

and

$$B = \frac{b}{2\sqrt{1-x_0}(1-\sqrt{1-x_0})} = b\frac{1-w_0}{1-2w_0}$$

$s = \dfrac{1}{2}$

x_0	w_0	$a = A$	b	B
$\frac{1}{4}$	$\frac{1}{2}-\frac{\sqrt{3}}{4}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{3+2\sqrt{3}}{4}$
$\frac{1}{64}$	$\frac{1}{2}-\frac{3\sqrt{7}}{16}$	$\frac{5}{16}$	$\frac{21}{8}$	$\frac{21+8\sqrt{7}}{16}$
-1	$\frac{1}{2}-\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	2	$\frac{2+\sqrt{2}}{2}$
$-\frac{1}{8}$	$\frac{1}{2}-\frac{3\sqrt{2}}{8}$	$\frac{\sqrt{2}}{4}$	$\frac{3\sqrt{2}}{2}$	$\frac{4+3\sqrt{2}}{4}$

$s = \dfrac{1}{3}$

x_0	w_0	$a = A$	b	B
$\frac{1}{2}$	$\dfrac{1}{2} - \dfrac{\sqrt{2}}{4}$	$\dfrac{\sqrt{3}}{9}$	$\dfrac{2\sqrt{3}}{3}$	$\dfrac{\sqrt{3}+\sqrt{6}}{9}$
$\dfrac{2}{27}$	$\dfrac{1}{2} - \dfrac{5\sqrt{3}}{18}$	$\dfrac{8}{27}$	$\dfrac{20}{9}$	$\dfrac{10+6\sqrt{3}}{9}$
$\dfrac{4}{125}$	$\dfrac{1}{2} - \dfrac{11\sqrt{5}}{50}$	$\dfrac{8\sqrt{3}}{45}$	$\dfrac{22\sqrt{3}}{45}$	$\dfrac{11\sqrt{3}+5\sqrt{15}}{15}$
$-\dfrac{9}{16}$	$-\dfrac{1}{8}$	$\dfrac{\sqrt{3}}{4}$	$\dfrac{5\sqrt{3}}{4}$	$\dfrac{9\sqrt{3}}{8}$
$-\dfrac{1}{16}$	$\dfrac{1}{2} - \dfrac{\sqrt{17}}{8}$	$\dfrac{7\sqrt{3}}{36}$	$\dfrac{17\sqrt{3}}{12}$	$\dfrac{17\sqrt{3}+4\sqrt{51}}{24}$
$-\dfrac{1}{80}$	$\dfrac{1}{2} - \dfrac{9\sqrt{5}}{40}$	$\dfrac{\sqrt{15}}{12}$	$\dfrac{3\sqrt{15}}{4}$	$\dfrac{20\sqrt{3}+9\sqrt{15}}{24}$
$-\dfrac{1}{1024}$	$\dfrac{1}{2} - \dfrac{5\sqrt{41}}{64}$	$\dfrac{53\sqrt{3}}{288}$	$\dfrac{205\sqrt{3}}{96}$	$\dfrac{205\sqrt{3}+32\sqrt{123}}{192}$
$-\dfrac{1}{3024}$	$\dfrac{1}{2} - \dfrac{55\sqrt{21}}{504}$	$\dfrac{13\sqrt{7}}{108}$	$\dfrac{55\sqrt{7}}{36}$	$\dfrac{84\sqrt{3}+55\sqrt{7}}{72}$
$-\dfrac{1}{500^2}$	$\dfrac{1}{2} - \dfrac{53\sqrt{89}}{1000}$	$\dfrac{827\sqrt{3}}{4500}$	$\dfrac{4717\sqrt{3}}{1500}$	$\dfrac{4717\sqrt{3}+500\sqrt{267}}{3000}$

$s = \frac{1}{4}$

x_0	w_0	$a = A$	b	B
$\frac{32}{81}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{14}{9}$	$\frac{16}{9}$
$\frac{1}{9}$	$\frac{1}{2} - \frac{\sqrt{2}}{3}$	$\frac{\sqrt{3}}{6}$	$\frac{4\sqrt{3}}{3}$	$\frac{4\sqrt{3}+3\sqrt{6}}{6}$
$\frac{1}{81}$	$\frac{1}{2} - \frac{2\sqrt{5}}{9}$	$\frac{2\sqrt{2}}{9}$	$\frac{20\sqrt{2}}{9}$	$\frac{20\sqrt{2}+9\sqrt{10}}{18}$
$\frac{1}{49^2}$	$\frac{1}{2} - \frac{10\sqrt{2}}{49}$	$\frac{9\sqrt{3}}{49}$	$\frac{120\sqrt{3}}{49}$	$\frac{147\sqrt{2}+120\sqrt{3}}{98}$
$\frac{1}{99^2}$	$\frac{1}{2} - \frac{35\sqrt{2}}{99}$	$\frac{19\sqrt{11}}{198}$	$\frac{140\sqrt{11}}{99}$	$\frac{140\sqrt{11}+99\sqrt{22}}{198}$
$\frac{1}{99^4}$	$\frac{1}{2} - \frac{910\sqrt{29}}{9801}$	$\frac{2206\sqrt{2}}{9801}$	$\frac{52780\sqrt{2}}{9801}$	$\frac{52780\sqrt{2}+9801\sqrt{58}}{19602}$
$-\frac{1}{4}$	$\frac{1}{2} - \frac{\sqrt{5}}{4}$	$\frac{3}{8}$	$\frac{5}{2}$	$\frac{5+2\sqrt{5}}{4}$
$-(\frac{16}{63})^2$	$-\frac{1}{63}$	$\frac{8\sqrt{7}}{63}$	$\frac{65\sqrt{7}}{63}$	$\frac{64\sqrt{7}}{63}$
$-\frac{1}{48}$	$\frac{1}{2} - \frac{7\sqrt{3}}{24}$	$\frac{3\sqrt{3}}{16}$	$\frac{7\sqrt{3}}{4}$	$\frac{12+7\sqrt{3}}{8}$
$-\frac{1}{324}$	$\frac{1}{2} - \frac{5\sqrt{13}}{36}$	$\frac{23}{72}$	$\frac{65}{18}$	$\frac{65+18\sqrt{13}}{36}$
$-\frac{1}{5\cdot 72^2}$	$\frac{1}{2} - \frac{161\sqrt{5}}{720}$	$\frac{41\sqrt{5}}{288}$	$\frac{161\sqrt{5}}{72}$	$\frac{360+161\sqrt{5}}{144}$
$-\frac{1}{882^2}$	$\frac{1}{2} - \frac{145\sqrt{37}}{1764}$	$\frac{1123}{3528}$	$\frac{5365}{882}$	$\frac{5365+882\sqrt{37}}{1764}$

$s = \frac{1}{6}$

x_0	w_0	$a = A$	b	B
$\frac{27}{125}$	$\frac{1}{2} - \frac{7\sqrt{10}}{50}$	$\frac{3\sqrt{5}}{25}$	$\frac{28\sqrt{5}}{25}$	$\frac{25\sqrt{2}+14\sqrt{5}}{25}$
$\frac{4}{125}$	$\frac{1}{2} - \frac{11\sqrt{5}}{50}$	$\frac{2\sqrt{15}}{25}$	$\frac{22\sqrt{15}}{25}$	$\frac{25\sqrt{3}+11\sqrt{15}}{25}$
$\frac{8}{11^3}$	$\frac{1}{2} - \frac{21\sqrt{33}}{242}$	$\frac{20\sqrt{33}}{363}$	$\frac{84\sqrt{33}}{121}$	$2 + \frac{42\sqrt{33}}{121}$
$\frac{64}{85^3}$	$\frac{1}{2} - \frac{171\sqrt{1785}}{14450}$	$\frac{144\sqrt{255}}{7225}$	$\frac{2394\sqrt{255}}{7225}$	$\sqrt{7} + \frac{1197\sqrt{255}}{7225}$
$-(\frac{4}{5})^3$	$\frac{1}{2} - \frac{3\sqrt{105}}{50}$	$\frac{8\sqrt{15}}{75}$	$\frac{21\sqrt{15}}{25}$	$\frac{25\sqrt{7}+21\sqrt{15}}{50}$
$-(\frac{3}{8})^3$	$\frac{1}{2} - \frac{7\sqrt{22}}{64}$	$\frac{15\sqrt{2}}{64}$	$\frac{77\sqrt{2}}{32}$	$\frac{77\sqrt{2}+32\sqrt{11}}{64}$
$-\frac{1}{8^3}$	$\frac{1}{2} - \frac{3\sqrt{114}}{64}$	$\frac{25\sqrt{6}}{192}$	$\frac{57\sqrt{6}}{32}$	$\frac{57\sqrt{6}+32\sqrt{19}}{64}$
$-\frac{9}{40^3}$	$\frac{1}{2} - \frac{253\sqrt{10}}{1600}$	$\frac{93\sqrt{30}}{1600}$	$\frac{759\sqrt{30}}{800}$	$\frac{2400\sqrt{3}+759\sqrt{30}}{1600}$
$-\frac{1}{80^3}$	$\frac{1}{2} - \frac{63\sqrt{645}}{3200}$	$\frac{263\sqrt{15}}{3200}$	$\frac{2709\sqrt{15}}{1600}$	$\frac{2709\sqrt{15}+1600\sqrt{43}}{3200}$
$-\frac{1}{440^3}$	$\frac{1}{2} - \frac{651\sqrt{22110}}{193600}$	$\frac{10177\sqrt{330}}{580800}$	$\frac{43617\sqrt{330}}{96800}$	$\frac{96800\sqrt{67}+43617\sqrt{330}}{193600}$
$-\frac{1}{53360^3}$	$\frac{1}{2} - \frac{651\sqrt{22110}}{193600}$	$\frac{13591409\sqrt{10005}}{227897059584000}$	$\frac{90856689\sqrt{10005}}{37982843264000}$	see below

In the last row

$$B = \frac{711822400\sqrt{163} + 90856689\sqrt{10005}}{75965686528000}$$

Remark: When x_0 is positive then we get a (slowly) convergent series with $\frac{1}{2}(1+\sqrt{1-x_0})$ but the sum is not $\frac{1}{\pi}$ (rather a negative multiple of it). Why?

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Appendix: A class of slowly converging series for $1/\pi$.

Arne Meurman

In the final remark Almkvist and Aycock ask why, when one considers the power series at $w_1 = \frac{1}{2}(1+\sqrt{1-x_0})$, instead of at $w_0 = \frac{1}{2}(1-\sqrt{1-x_0})$, one gets formulas for negative multiples of $\frac{1}{\pi}$. Here we shall prove such formulas in the cases $s = 1/2, 1/3, 1/4, 1/6$ in Proposition 7.

Following [2] we set

$$F(t) = F(s,t) = {}_2F_1(s, 1-s; 1; t),$$

$$G(t) = t\frac{dF}{dt},$$

and let for $s = 1/2, 1/3, 1/4, 1/6$ $t(\tau) = t_N(\tau)$ be given by

$$t_4(\tau) = \left(1 + \frac{1}{16}\left(\frac{\eta(\tau)}{\eta(4\tau)}\right)^8\right)^{-1}, \quad t_3(\tau) = \left(1 + \frac{1}{27}\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{12}\right)^{-1},$$

$$t_2(\tau) = \left(1 + \frac{1}{64}\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24}\right)^{-1}, \quad t_1(\tau) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{1728}{j(\tau)}}.$$

Let U be the connected component of $\{\tau \in \mathbf{C} \mid \Im(\tau) > 0, |t(\tau)| < 1\}$ which contains all τ with sufficiently large imaginary part, a "neighborhood of $i\infty$". Let $\tau_0 \in U$ such that

$$w_0 = t(\tau_0), \quad w_1 = 1 - t(\tau_0) \tag{1}$$

satisfy

$$|w_0| < 1, \quad |w_1| < 1. \tag{2}$$

In the Ramanujan-type formulas, τ_0 is usually a quadratic irrationality. Let A_n be defined by the power series expansions

$$F^2(t) = \sum_{n=0}^{\infty} A_n t^n, \tag{3}$$

as in Proposition 7. Set

$$C_s = \frac{1}{2\sin(\pi s)}. \tag{4}$$

Theorem 1 *Assume that there is an identity*

$$\sum_{n=0}^{\infty}(A + Bn)A_n w_0^n = \frac{C}{\pi}, \tag{5}$$

equivalently

$$AF^2(w_0) + 2BF(w_0)G(w_0) = \frac{C}{\pi}. \tag{6}$$

Then

$$\sum_{n=0}^{\infty}(\hat{A}+\hat{B}n)A_n w_1^n = \frac{\hat{C}}{\pi}, \tag{7}$$

where

$$\begin{aligned}\hat{A} &= A,\\ \hat{B} &= -B\frac{w_0}{w_1},\\ \hat{C} &= \frac{C\left(\frac{\tau_0}{i}\right)^2}{C_s^2} - \frac{B\left(\frac{\tau_0}{i}\right)}{C_s^2 w_1}.\end{aligned} \tag{8}$$

Proof. By formulas (8), (9) in [2] we have, for $\tau \in U$,

$$\tau = iC_s\frac{F(1-t)}{F(t)}, \tag{9}$$

and

$$\frac{1}{2\pi i}\frac{dt}{d\tau} = q\frac{dt}{dq} = t(1-t)F^2(t), \tag{10}$$

where $q = e^{2\pi i\tau}$. Take $\frac{1}{2\pi i}$ times the logarithmic derivative with respect to τ in (9):

$$\frac{\frac{dF}{dt}(1-t)\cdot(-1)\cdot\frac{1}{2\pi i}\frac{dt}{d\tau}}{F(1-t)} - \frac{\frac{dF}{dt}\frac{1}{2\pi i}\frac{dt}{d\tau}}{F(t)} = \frac{1}{2\pi i\tau}.$$

Substitute (10) to obtain

$$-\frac{G(1-t)tF^2(t)}{F(1-t)} - G(t)(1-t)F(t) = \frac{1}{2\pi i\tau}.$$

Multiply by $2i\tau$ and substitute (9):

$$2C_s\, t\, G(1-t)F(t) + 2\left(\frac{\tau}{i}\right)(1-t)G(t)F(t) = \frac{1}{\pi}. \tag{11}$$

Evaluate (11) at $\tau = \tau_0$:

$$2C_s\, w_0\, G(w_1)F(w_0) + 2\left(\frac{\tau_0}{i}\right)w_1 G(w_0)F(w_0) = \frac{1}{\pi}. \tag{12}$$

By assumption (6)

$$2BF(w_0)G(w_0) = \frac{C}{\pi} - AF^2(w_0), \tag{13}$$

and eliminating $F(w_0)G(w_0)$ by (12), (13) we have

$$2BC_s\, w_0\, G(w_1)F(w_0) - A\left(\frac{\tau_0}{i}\right)w_1F^2(w_0) = \frac{1}{\pi}\left(B - C\left(\frac{\tau_0}{i}\right)w_1\right).$$

Substitute $F(w_0) = \left(\frac{i}{\tau_0}\right)C_sF(w_1)$ from (9) to obtain

$$-A\left(\frac{i}{\tau_0}\right)C_s^2 w_1F^2(w_1) + 2B\left(\frac{i}{\tau_0}\right)C_s^2\, w_0\, F(w_1)G(w_1) = \frac{1}{\pi}\left(B - C\left(\frac{\tau_0}{i}\right)w_1\right).$$

Dividing by $-\left(\frac{i}{\tau_0}\right) C_s^2 w_1$ we get

$$AF^2(w_1) - 2B\frac{w_0}{w_1}F(w_1)G(w_1) = \frac{1}{\pi}\left(\frac{C\left(\frac{\tau_0}{i}\right)^2}{C_s^2} - \frac{B\left(\frac{\tau_0}{i}\right)}{C_s^2 w_1}\right),$$

which is equivalent to (7). ■

Remark. The arguments in the proof above are analogous to some arguments in the proof of [1] Theorem 2.1.

Theorem 1 applies to all of the identities in the Tables following Proposition 7 where $x_0 > 0$. We present 4 examples of such.

Example 1. In case $s = \frac{1}{4}, \tau_0 = i$ we have $w_0 = t_2(i) = \frac{1}{9}$, and there is in [4], (1.12) the identity

$$\sum_{n=0}^{\infty}(1+8n)A_n\frac{1}{9^n} = \frac{9}{2\pi}, \tag{14}$$

where

$$A_n = \frac{1}{64^n}\sum_{k=0}^{n}\binom{2k}{k}\binom{4k}{2k}\binom{2(n-k)}{n-k}\binom{4(n-k)}{2(n-k)}. \tag{15}$$

In this case

$$A = 1, \quad B = 8, \quad C = \frac{9}{2}, \quad C_s = \frac{1}{\sqrt{2}},$$

and we obtain

$$w_1 = \frac{8}{9},$$

$$\hat{A} = 1, \quad \hat{B} = -1, \quad \hat{C} = -9,$$

$$\sum_{n=0}^{\infty}(1-n)A_n\left(\frac{8}{9}\right)^n = -\frac{9}{\pi}, \tag{16}$$

which proves [3], (2.10).

Example 2. In case $s = \frac{1}{4}, \tau_0 = \frac{\sqrt{58}}{2}i$ we have

$$w_0 = t_2\left(\frac{\sqrt{58}}{2}i\right) = \frac{1}{2} - \frac{910}{9801}\sqrt{29},$$

and there is in Proposition 7, Table $s = \frac{1}{4}$, the identity

$$\sum_{n=0}^{\infty}\left(\frac{2206\sqrt{2}}{9801} + \frac{52780\sqrt{2}+9801\sqrt{58}}{19602}n\right)A_n w_0^n = \frac{1}{\pi}, \tag{17}$$

where A_n is as in (15). In this case

$$C_s = \frac{1}{\sqrt{2}}, \quad w_1 = \frac{1}{2} + \frac{910}{9801}\sqrt{29},$$

and we obtain

$$\sum_{n=0}^{\infty}\left(\frac{2206\sqrt{2}}{9801} + \frac{52780\sqrt{2}-9801\sqrt{58}}{19602}n\right)A_n w_1^n = -\frac{29}{\pi}. \tag{18}$$

Example 3. In case $s=\frac{1}{2}, \tau_0=\frac{\sqrt{3}}{2}i$ we have

$$w_0=t_4\left(\frac{\sqrt{3}}{2}i\right)=\frac{1}{2}-\frac{\sqrt{3}}{4},$$

and there is in Proposition 7, Table $s=\frac{1}{2}$, the identity

$$\sum_{n=0}^{\infty}\left(\frac{1}{4}+\frac{3+2\sqrt{3}}{4}n\right)A_n w_0^n=\frac{1}{\pi}, \tag{19}$$

where

$$A_n=\frac{1}{16^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2. \tag{20}$$

In this case

$$C_s=\frac{1}{2}, \quad w_1=\frac{1}{2}+\frac{\sqrt{3}}{4},$$

and we obtain

$$\sum_{n=0}^{\infty}\left(\frac{1}{4}+\frac{3-2\sqrt{3}}{4}n\right)A_n w_1^n=-\frac{3}{\pi}. \tag{21}$$

Example 4. In case $s=\frac{1}{6}, \tau_0=\sqrt{7}i$ we have

$$w_0=t_1\left(\sqrt{7}i\right)=\frac{1}{2}-\frac{171}{14450}\sqrt{1785},$$

and there is in Proposition 7, Table $s=\frac{1}{6}$ the identity

$$\sum_{n=0}^{\infty}(A+Bn)A_n w_0^n=\frac{1}{\pi}, \tag{22}$$

where

$$\begin{aligned} A&=\frac{144\sqrt{255}}{7225}, \quad B=\sqrt{7}+\frac{1197\sqrt{255}}{7225}, \\ A_n&=\frac{1}{432^n}\sum_{k=0}^{n}\binom{3k}{k}\binom{6k}{3k}\binom{3(n-k)}{n-k}\binom{6(n-k)}{3(n-k)}. \end{aligned} \tag{23}$$

In this case

$$C_s=1, \quad w_1=\frac{1}{2}+\frac{171}{14450}\sqrt{1785},$$

and we obtain

$$\sum_{n=0}^{\infty}(\hat{A}+\hat{B}n)A_n w_1^n=-\frac{7}{\pi}, \tag{24}$$

where

$$\hat{A}=A, \quad \hat{B}=-\sqrt{7}+\frac{1197\sqrt{255}}{7225}.$$

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